DIFFERENTIAL TOPOLOGY

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1. Introduction

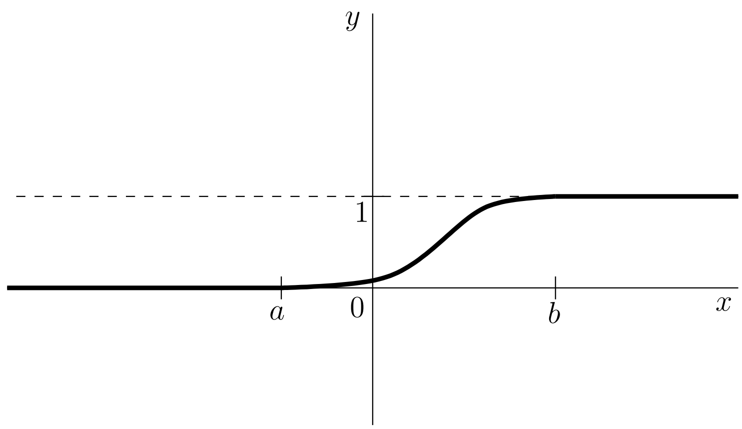
This paper is to propose solutions to selected exercises in Differential Topology by Guillemin and Pollack, [1], and to comment on certain proofs in the book. Although the sections covered in this paper correspond to specific sections in the book, they have been freely renamed to suit the content. The sections of the book which will be covered here are, in order:

§6: Homotopy and Stability
Appendix 1.: Measure Zero and Sard's Theorem
§8: Embedding Manifolds in Euclidean Space
Appendix 2.: Classification of Compact One-Manifolds
§2: One-Manifolds and Some Consequences

Unless specified, references to sections (denoted by the symbol §) are to those in this paper.

2. A Smooth Function

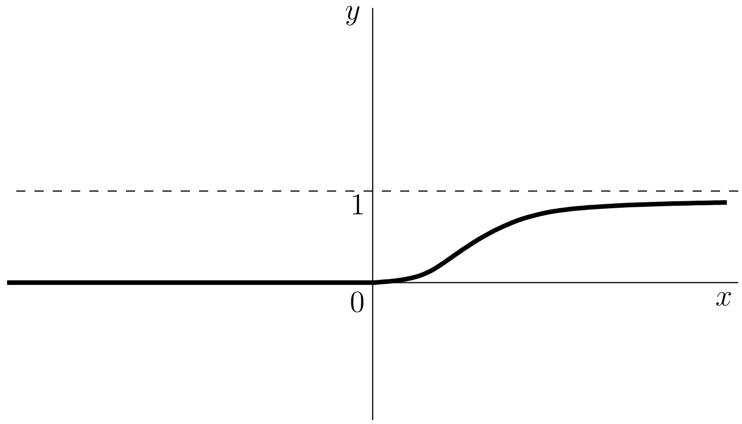
The exercises involving homotopy can be solved more easily by making use of a smooth function which is identically zero before a certain point $a \in \mathbf{R}$ and identically one after a certain point $b \in \mathbf{R}$. The graph of this function looks more or less like this:



Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Date: September 23, 2005.
This paper was prepared for a Research Experience for the Undergraduates in the field of differential topology with Professor Geoffrey Mess at the University of California, Los Angeles in Summer 2005.



Lemma 1. *f is a smooth function.*

Proof. f is clearly smooth when $x < 0$, since all of its derivatives are then zero. When $x > 0$,

$$f^{(n)}(x) = P_{n+1} x^{-2n} e^{-\frac{1}{x}} \quad \text{for all } n \geq 1, \tag{1}$$

where P_n is a polynomial in x of degree n. For

$$f'(x) = \frac{1}{x^2} e^{-\frac{1}{x}} \quad \text{for all } x > 0,$$

and if we assume that (1) holds for the n-th derivative of f, then

$$f^{(n+1)}(x) = \left[P'_{n+1} x^2 - 2n P_{n+1} x + P_{n+1}\right] x^{-2(n+1)} e^{-\frac{1}{x}}.$$

Since the polynomial inside the square brackets has degree $n+2$, this proves (1) by induction. It remains to be shown that f is smooth at the origin. In fact, $f^{(n)}(0) = 0$ for all $n \in \mathbf{N}$. Indeed, $f(0) = 0$ is differentiable and its derivative is given by

$$\lim_{h \searrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \searrow 0} \frac{1}{h} e^{-\frac{1}{h}} = \lim_{u \to +\infty} u e^{-u} = \lim_{u \to +\infty} \frac{1}{e^u},$$

where the last equality holds by L'Hospital's rule. Thus, the above expression equals zero and agrees with the limit approaching the origin from the negative real axis. Suppose now that $f^{(n-1)}$ is differentiable at 0, with $f^{(n)}(0) = 0$. If P_{n+1} is defined as in (1), we have $P_{n+1}(x) = \sum_{i=0}^{n+1} \alpha_i x^i$ whenever $x > 0$ for some $\alpha_i \in \mathbf{R}$, $0 \leq i \leq n+1$ and α_0 non-zero. Then

$$\begin{aligned}
\lim_{h \searrow 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} &= \lim_{h \searrow 0} \frac{P_{n+1}(h) h^{-2n-1} e^{-\frac{1}{h}}}{h} \\
&= \lim_{h \searrow 0} \alpha_0 h^{-2n-2} \left(1 + \sum_{i=1}^{n+1} \frac{\alpha_i}{\alpha_0} h^i\right) e^{-\frac{1}{h}} \\
&= \lim_{h \searrow 0} \alpha_0 h^{-2n-2} e^{-\frac{1}{h}} \\
&= \lim_{u \to +\infty} \alpha_0 u^{2n+2} e^{-u}.
\end{aligned}$$

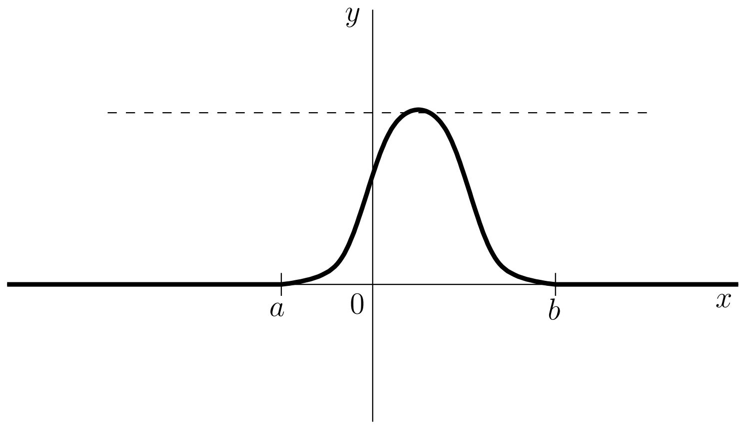
Again by L'Hospital's rule, the above expression equals

$$\lim_{u \to +\infty} \alpha_0 (2n+2)! e^{-u} = 0.$$

This limit agrees with that coming from the left which is naturally zero, so that $f^{(n)}$ is differentiable at 0 and $f^{(n+1)} = 0$. $\square$

Now we may construct the smooth function $E_a^b : \mathbf{R} \to \mathbf{I} = [0,1]$ such that it is identically zero when $x \leq a$ and identically one when $x \geq b$ for some real numbers a and b which satisfy $a < b$. This is easily achieved by integrating the "bump function" $\beta : \mathbf{R} \to \mathbf{I}$

$$\beta(x) = \begin{cases} e^{-\frac{1}{x-a}} e^{\frac{1}{x-b}} & a < x < b \\ 0 & \text{otherwise} \end{cases},$$



and then normalizing it. More precisely we have

$$E_a^b(x) = \frac{\int_a^x \beta(t)dt}{\int_a^b \beta(t)dt}.$$

Let us verify that this proposed construction of E_a^b actually satisfies the smoothness requirement. β is the product of two smooth functions on $[a,b]$ and therefore is smooth itself. Its integral is a continuous function with smooth derivative, for which we conclude that E_a^b is smooth.

1. Suppose that $f_0, f_1 : X \to Y$ are homotopic. There exists a smooth function $F : X \times \mathbf{I} \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Consider the smooth function $\widetilde{F} : X \times \mathbf{I} \to Y$ by

$$\widetilde{F}(x,t) = F\left(x, E_{1/4}^{3/4}(t)\right).$$

   $\widetilde{F}$ is smooth because it is composed of smooth functions. Moreover, $\widetilde{F}(x,t) = f_0(x)$ if $t \leq \frac{1}{4}$, and $\widetilde{F} = f_1(x)$ if $t \geq \frac{3}{4}$. Thus $\widetilde{F}$ is another homotopy from f_0 to f_1.
2. Now we show that homotopy is an equivalence relation. For any smooth map $f : X \to Y$, $f \sim f$. For a homotopy is $F : X \times \mathbf{I} \to Y$ by $F(x,t) = f(x)$. Homotopy is also a symmetric property. If $f \sim g$ by some homotopy G satisfying $G(x,0) = f(x)$ and $G(x,1) = g(x)$, then $(x,t) \mapsto G(x, 1-t)$ is a homotopy from g to f. Finally, suppose $f \sim g$ and $g \sim h$ by homotopies H and J respectively. Then $f \sim h$ by

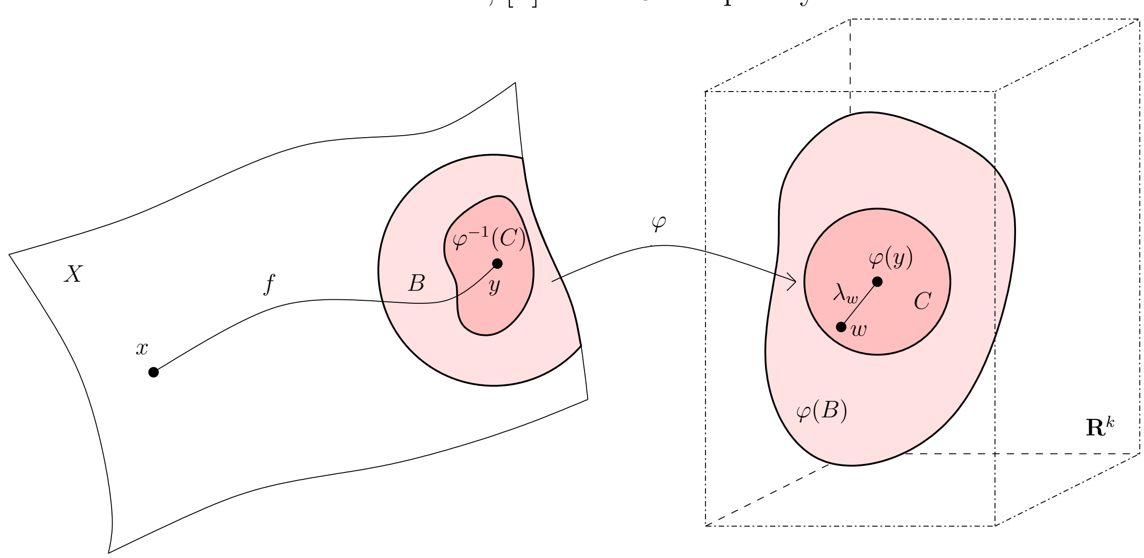
$$(x,t) \mapsto \begin{cases} H\left(x, E_{1/6}^{1/3}(t)\right) & 0 \leq t \leq \frac{1}{2} \\ J\left(x, E_{2/3}^{5/6}(t)\right) & \frac{1}{2} < t \leq 1 \end{cases},$$

a smooth function by the previous problem.

3. Let us show that any connected manifold X is arcwise connected. We begin by proving that "x_0 and x_1 can be joined by a smooth curve" is an equivalence relation. This is true because $x_0, x_1 \in X$ satisfy the relation if and only if there is a smooth function $f : \mathbf{I} \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. Now for an arbitrary manifold Y, let $F : Y \times \mathbf{I} \to X$ by $F(x,t) = f(t)$. So $F(x,0) = x_0$ and $F(x,1) = x_1$. In other words $x_0, x_1 \in Y$ satisfy the relation if and only if there is a homotopy between the constant functions $F(x,0)$ and $F(x,1)$. But homotopy is an equivalence relation by the previous problem. So "x_0 and x_1 can be joined by a smooth curve" is an equivalence relation as well.

   Define the equivalence class of $x \in X$ by
$$[x] = \{y \in X | \text{“}x \text{ and } y \text{ can be joined by a smooth curve”}\}.$$
   We claim that such a set is open. Suppose that $y \in [x]$, so that there is a smooth function $f : \mathbf{I} \to X$ with $f(0) = x$ and $f(1) = y$. Since X is a manifold, there exists near y a local diffeomorphism $\varphi : B \to \mathbf{R}^k$, where B is a sufficiently small neighborhood of y open in X, and k is the dimension of X, such that $\varphi(B)$ is open in $\mathbf{R}^k$. Note that $\varphi(y)$ belongs to $\varphi(B)$, and choose an open ball C contained in $\varphi(B)$ centered at y. Then C is arcwise connected, because every point w in C and $\varphi(y)$ can be joined by a straight line $\lambda_w : \mathbf{I} \to C$ defined by $\lambda_w(t) = (1-t)\varphi(y) + tw$. By the continuity of φ, the preimage of C, denoted $\varphi^{-1}(C)$, is open. It is also easily seen that $\varphi^{-1}(C)$ is arcwise connected. Let $z \in \varphi^{-1}(C)$ and define $\psi : \mathbf{I} \to \varphi^{-1}(C)$ by $\psi = \varphi^{-1} \circ \lambda_{\varphi(z)}$. The image of ψ is contained in $\varphi^{-1}(C)$, since the image of $\lambda_{\varphi(z)}$ is a subset of C. Moreover $\psi(0) = y$ and $\psi(1) = z$, so $z \in [y] = [x]$. Therefore $[x]$ is open.

   Now the complement of $[x]$ is the union of all the other equivalence classes, i.e. it is a union of open sets, and so must be open as well. Hence $[x]$ is closed. Since $[x]$ is a subset of the connected manifold X, $[x] = X$. Consequently X is arcwise connected.



4. Suppose that X is a contractible manifold. We show that all maps from an arbitrary manifold Y into X are homotopic. Because X is contractible, there is a smooth map $F : X \times \mathbf{I} \to X$ such that $F(x,0) = x$ and $F(x,1) = z$ for a particular $z \in X$. Let g

and h be two maps from Y into X, and define $G : Y \times \mathbf{I} \to X$ by

$$G(x,t) = \begin{cases} F\left(g(x), E_0^{1/3}(t)\right) & 0 \le t \le \frac{1}{2} \\ F\left(h(x), 1 - E_{2/3}^1(t)\right) & \frac{1}{2} < t \le 1 \end{cases}.$$

G is smooth since $G(x,t) = z$ when $\frac{1}{3} \le t \le \frac{2}{3}$. Also, $G(x,0) = F(g(x),0) = g(x)$ and $G(x,1) = F(h(x),0) = h(x)$. So G is a homotopy between g and h.

5. We now show that $\mathbf{R}^k$ is contractible. Let $y \in \mathbf{R}^k$ and define $F : \mathbf{R}^k \times \mathbf{I} \to \mathbf{R}^k$ by

$$F(x,t) = tx + (1-t)y.$$

F is smooth, with $F(x,0) = y$ and $F(x,1) = x = \mathrm{Id}(x)$. So F is contractible.

6. We claim that any contractible manifold X is simply connected. By problem (4), all maps from the manifold S^1 into X are homotopic. In particular they are homotopic to any constant. If it can also be shown that X is connected, then X is simply connected. Since X is contractible, there is by definition a homotopy f from the identity map on X to a constant $z \in X$. But then, any $x \in X$ can be joined to z by a path $f_x : \mathbf{I} \to X$ defined by $f_x(t) = f(x,t)$. This implies that X is arcwise connected, hence connected.

7. The antipodal map $\alpha_k : S^k \to S^k$ by $x \mapsto -x$ is homotopic to the identity if k is odd. We can see the statement holds for $k = 1$ by considering S^1 as a subset of $\mathbf{C}$. Then $S^1 \times \mathbf{I} \to \mathbf{C}$ by $(x,t) \mapsto e^{i\pi t}x$ is a suitable homotopy between α_1 and the identity on S^1, for the image of the homotopy is contained in S^1. Suppose now that S^{2k-1} lies in $\mathbf{C}^k$, for some positive integer k. The map $S^{2k-1} \times \mathbf{I} \to \mathbf{C}^k$ by

$$(z,t) \mapsto e^{i\pi t} z = e^{i\pi t} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{pmatrix}$$

joins α_k to the identity on S^{2k-1} and maps into S^{2k-1} since $|e^{i\pi t}x| = 1$. Hence the constructed function is our desired homotopy.

3. Measure zero and Sard's Theorem

Several proofs for results which were mentioned in Appendix 1 of [1] were left as exercises to the reader. Among them are the facts that the open solids in the definition of a set of measure zero can be taken to be cubes, and that $\mathbf{R}^{n-1}$ has measure zero in $\mathbf{R}^n$. In this section, we prove these results and the Mini-Sard theorem. Then we attempt to clarify the proof of Sard's Theorem given by [1].

Theorem 2. *The natural copy of* $\mathbf{R}^{n-1}$ *in* $\mathbf{R}^n$ *has measure zero.*

Proof. Since $\mathbf{R}^{n-1}$ has a natural copy in $\mathbf{R}^n$, we will assume that $x \in \mathbf{R}^{n-1}$, if it has the form $x = (x_1, x_2, \ldots x_{n-1}, 0)$ for some $x_1, \ldots x_{n-1} \in \mathbf{R}$, and take the definitions in [1] of a rectangular solid

$$S(a,b) = \{x \in \mathbf{R}^n | a_i < x_i < b_i, 1 \le i \le n\},$$

and of the volume of a rectangular solid,

$$\operatorname{vol}(S(a,b)) = \prod_{i=1}^{n}(b_i - a_i).$$

Given an $\varepsilon > 0$ define a sequence $\{\delta_i\}_{i=1}^{\infty}$ so that

$$0 < \delta_i < 2^{-(i+1)}(1.01)^{1-n}\varepsilon.$$

The rectangular solids $S'(a', b') \subset \mathbf{R}^{n-1}$, such that $a' = (a_1, \dots a_{n-1})$, $b' = (a_1+1.01, \dots a_{n-1}+1.01)$, and each a_i is an integer, form a countable set. Let $\{A'_i\}_{i=1}^{\infty}$ be an enumeration of these and let $A_i = S(a,b) \subset \mathbf{R}^{n-1}$ where $a = (a_1, \dots a_{n-1}, -\delta_i)$ and $b = (b_1, \dots b_{n-1}, \delta)$. Then,

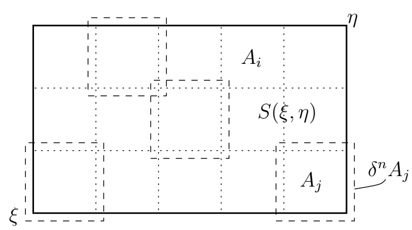
$$\sum_{i=1}^{\infty} \operatorname{vol}(A_i) = \sum_{i=1}^{\infty} 1.01^{n-1} 2\delta_i < \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \varepsilon.$$

So $\mathbf{R}^{n-1}$ has measure zero in $\mathbf{R}^n$. □

Lemma 3. *A solid $S(\xi, \eta)$, $\xi = (\xi_1, \dots \xi_n)$ and $\eta = (\eta_1, \dots \eta_n)$, whose vertices have rational coordinates and whose volume is strictly less than some $\varepsilon > 0$ can be covered by finitely many cubes whose total volume is still less than ε.*

Proof. For any i, $\eta_i - \xi_i \in \mathbf{Q}^+$, there exists an $\alpha_i \in \mathbf{Q}^+$ and a common denominator $\beta \in \mathbf{Q}^+$ for all i so that $\eta_i - \xi_i = \alpha_i/\beta$. Hence, $S(\xi, \eta)$ can be divided into $\prod_{i=1}^{n} \alpha_i$ semi-open cubes (which include some of its boundary faces) with sides $1/\beta$ and volume β^{-n}. The open cubes A_j obtained by ignoring all the boundaries of the semi-open cubes occupy a volume equal to that of the original $S(\xi, \eta)$ although they do not cover it. This can be compensated, however, by expanding each A_j by a factor $\delta > 1$ about its center, such that δ satisfies

$$\delta^n \sum_{j=1}^{\prod \alpha_i} \operatorname{vol}(A_j) = \delta^n \operatorname{vol} S(\xi, \eta) < \varepsilon.$$



Thus, the expanded cubes form a cover for $S(\xi, \eta)$ with the properties required by the Lemma. □

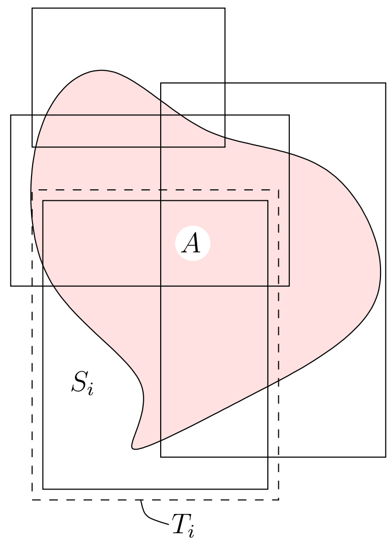
By definition, a set in $\mathbf{R}^n$ has measure zero if it can be covered by countably many rectangular solids $S_1, S_2, \dots$ such that $\sum \operatorname{vol}(S_i) < \varepsilon$. We show that we can assume the solids are cubes $S(a,b)$, that is $b_1 - a_1 = \dots = b_n - a_n$.

Proof. Suppose that $A \in \mathbf{R}^n$ is a set of measure zero. Then for any $\varepsilon > 0$, there exists a covering $\{S_i\}_{i \in \mathbf{Z}^+}$ such that $\sum \operatorname{vol}(S_i) < \varepsilon$. Then there exist some real numbers ε_i for $i \in \mathbf{Z}^+$ such that $\operatorname{vol}(S_i) < \varepsilon_i$ and $\sum \varepsilon_i = \varepsilon$. Each $S_i = S(a_i, b_i)$ is contained in a rectangular solid

$$T_i = \prod_{j=1}^{n}(c_{ij}, d_{ij})$$

whose vertices have rational coordinates, and whose volume is less than ε_i. This claim can be made explicit by considering

$$\varepsilon_{ij} = \left(\frac{\varepsilon_i}{\prod_{k=1}^{n}(b_{ik} - a_{ik})} \right)^{1/n} (b_{ij} - a_{ij}) \text{ where } 1 \leq j \leq n.$$



Then $b_{ij} - a_{ij} < \varepsilon_{ij}$ and $\prod_{j=1}^{n} \varepsilon_{ij} = \varepsilon_i$. So there are $c_{ij}, d_{ij} \in \mathbf{Q}$ such that $a_{ij} - f_{ij} < c_{ij} < a_{ij}$ and $b_{ij} < d_{ij} < b_{ij} + f_{ij}$, where $f_{ij} = (\varepsilon_{ij} - b_{ij} + a_{ij})/2$.
By Lemma 3, T_i can be covered by finitely many cubes whose total volume is less than ε_i. So A is covered by countably cubes whose total volume is less than ε. □

The proof of the Mini-Sard theorem requires a theorem whose proof is given on page 204 of [1] and whose statement is as follows:

Theorem 4. *Let V be an open set of $\mathbf{R}^m$, and let $g : V \to \mathbf{R}^m$ be a smooth map. If $A \subset V$ is of measure zero, then $g(A)$ is of measure zero.*

Theorem 5. *(Mini-Sard) Let U be an open subset of $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^m$ be a smooth map. Then if $m > n$, $f(U)$ has measure zero in $\mathbf{R}^m$.*

Proof. Let $i : \mathbf{R}^n \to \mathbf{R}^m$ be the inclusion map. Then $i(U)$ is a subset of the natural copy of $\mathbf{R}^n$ in $\mathbf{R}^m$, which has measure zero by an induction on the claim above. ($\mathbf{R}^n$ has measure zero in $\mathbf{R}^{n+1}$, and if $\mathbf{R}^n$ has measure zero in $\mathbf{R}^m$, where $m > n$, then that $\mathbf{R}^m$ has measure zero in $\mathbf{R}^{m+1}$ implies that $\mathbf{R}^n$ is of measure zero in $\mathbf{R}^{m+1}$.) Therefore $i(U)$ has measure zero in $\mathbf{R}^m$. $f \circ i^{-1} : i(U) \to \mathbf{R}^m$ is a smooth map which can be locally extended to a smooth map on the open set $V = U \times \mathbf{R}^{m-n}$ containing $i(U)$, namely $f \circ \pi : V \to \mathbf{R}^m$, where π is the natural projection of $\mathbf{R}^m$ into $\mathbf{R}^n$. Theorem 4 guarantees that $f \circ \pi(i(U)) = f(U)$ has measure zero. □

Now are a few comments we thought ought to be made regarding the actual proof of Sard's Theorem in Appendix 1. The subsequent references to Lemmas 1 and 2 are to those in the proof of Sard's Theorem and not to those in this paper.

- A distinction should be made in the proof of Lemma 1 between the variable x of the map h and the point $\tilde{x}$ at which the derivative of h is taken. Therefore, we consider the points $\tilde{x} \in C \smallsetminus C_1$. A similar situation appears in the proof of Lemma 2, where we should distinguish the points $\tilde{x} \in C_k \smallsetminus C_{k+1}$ from the variable x of the function h.

- The double occurrence in the proof of Lemma 2 of the undefined expression "critical points of type C_k" may be found a bit puzzling by some. Most will however guess the meaning intended by the author, namely "points x of a function (in this case, g) whose partial derivatives of order less than or equal to k vanish at x". Nevertheless, some clarification should have been made, since C_k is a set defined for f only rather than a property enjoyed, in particular, by f.
- In Lemma 1, the definition of h holds because the fact that $\partial f/\partial x_1$ does not vanish at $\tilde{x}$ implies we can renumber the coordinate axes in $\mathbf{R}^p$ so that one of the components of f whose first partial derivative with respect to x_1 does not vanish becomes the first component of f.
- In Lemma 2, $h : U \to \mathbf{R}^n$ should be defined by $h(x) = (\rho_1(x), x_2, \dots x_n)$ where ρ_1 is a component of ρ whose first partial derivative $\partial\rho_1/\partial x_1$ does not vanish. We justify this claim by an argument analogous to the previous comment, for Lemma 1.
- It does not make sense to talk about the determinant of the matrix representing the derivative of g as it is not a square matrix. However, the intended meaning was that $\mathrm{Rank}((\partial g_i/\partial x_j)) = \mathrm{Rank}((\partial g_i^t/\partial x_j)) + 1$.

4. Embedding Manifolds

Several exercises in this section involve smooth maps f on a tangent bundle. It is necessary to use the global derivative map in order to study f locally in the Euclidean space. The reader will notice that we have named the global derivative Φ, rather than df as in [1], to simplify the notation when derivatives are taken. But first is a problem in §7 of [1].

6. The sphere S^k is simply connected. First, we show that it is connected. Any two distinct, non-antipodal points x and y on S^k are linearly independent, and hence define a plane P containing the origin. If x and y are antipodal, P can be defined by choosing an additional point on S^k distinct from either one of the antipodal pair. Let $f : \mathbf{I} \to P$ be a smooth function whose image is an arc of a circle centered at the origin with endpoints x and y. Since f is a path, S^k is arcwise connected, and consequently connected.
   Let $f : S^1 \to S^k$ be smooth. Then for all $x \in S^1$, $\mathrm{Rank}\, df_x \le 1 < k$. So S^1 contains only critical points of f. $f(S^1)$ consists entirely of critical values, so it is a subset of the set of all critical values of f which, by Sard's Theorem, has measure zero. Hence $f(S^1)$ has measure zero. As a result, we can choose a point $p \notin f(S^1)$. The image of $f(S^1)$ under the stereographic projection about p lies in a vector space isomorphic to $\mathbf{R}^k$. So denote by $\pi : S^k \smallsetminus \{p\} \to \mathbf{R}^k$ the composition of the stereographic projection with the isomorphism. By problem (5), §6 of [1], $\mathbf{R}^k$ is contractible, i.e. there is a homotopy $F : \mathbf{R}^k \times \mathbf{I} \to \mathbf{R}^k$ from the identity to a constant $z \in \mathbf{R}^k$. Define a new homotopy $\varphi : S^1 \times \mathbf{I} \to \mathbf{R}^k$ by
$$\varphi(x, t) = F(\pi \circ f(x), t).$$
   Then $\pi^{-1} \circ \varphi : S^1 \times \mathbf{I} \to S^k$ is a homotopy with the property that $\pi^{-1} \circ \varphi(x, 0) = f(x)$ and $\pi^{-1} \circ \varphi(x, 1) = \pi^{-1}(z)$, a constant in S^k. Therefore S^k is simply connected.

1.
$$T(\mathbf{R}^k) = \mathbf{R}^k \times \mathbf{R}^k.$$
To prove this statement, it suffices to show that for any $x \in \mathbf{R}^k$,
$$T_x(\mathbf{R}^k) = \mathbf{R}^k. \tag{2}$$
For, supposing that $\mathbf{R}^k$ lies in $\mathbf{R}^N$ for some $N \geq k$, we would then have that $T(\mathbf{R}^k) = \{(x,v) \in \mathbf{R}^k \times \mathbf{R}^N | v \in T_x(\mathbf{R}^k) = \mathbf{R}^k\} = \mathbf{R}^k \times \mathbf{R}^k$. Let φ be the inclusion map of $\mathbf{R}^k$ in $\mathbf{R}^N$. Then the matrix representation of $d\varphi_x$ is
$$\begin{pmatrix} I \\ 0 \end{pmatrix}$$
where I is the $k \times k$ identity matrix, and 0 is the $(n-k) \times k$ zero matrix. So $d\varphi_x$ is the inclusion map as well, and $T_x(\mathbf{R}^k) = \mathbf{R}^k$.

3. To show that $T(X \times Y)$ is diffeomorphic to $T(X) \times T(Y)$, we will first show that for any $(x,y) \in X \times Y$,
$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y). \tag{3}$$
Suppose that $\varphi : \mathbf{R}^k \to U$ and $\psi : \mathbf{R}^l \to V$ are parametrizations of neighborhoods $U \subset X$ and $V \subset Y$ around x and y respectively. If $\varphi(\xi) = x$ and $\psi(\eta) = y$, then for any $(v,w) \in \mathbf{R}^k \times \mathbf{R}^l$
$$\begin{aligned} d(\varphi \times \psi)_{(\xi,\eta)}(v,w) &= \lim_{t \to 0} \tfrac{1}{t}\big[(\varphi \times \psi)[(\xi,\eta) + t(v,w)] - (\varphi \times \psi)(\xi,\eta)\big] \\ &= \Big(\lim_{t\to 0} \tfrac{1}{t}[\varphi(\xi + tv) - \varphi(\xi)], \lim_{t\to 0} \tfrac{1}{t}[\psi(\eta + tw) - \psi(\eta)]\Big) = d\varphi_\xi(v) \times d\psi_\eta(w). \end{aligned}$$
By definition, the image of $d(\varphi \times \psi)_{(\xi,\eta)}$ is $T_{(x,y)}(X,Y)$ and that of $d\varphi_x \times d\psi_y$ is $T_x(X) \times T_y(Y)$. Therefore (3) holds.
Now define $f : T(X) \times T(Y) \to T(X \times Y)$ by $f((x,v),(y,w)) = ((x,y),(v,w))$. f is clearly smooth and injective. If $(z,u) \in T(X \times Y)$ where $z \in X \times Y$, then there exist $x \in X$ and $y \in Y$ such that $z = (x,y)$. Moreover, $u \in T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$, so that there exist $v \in T_x(X)$ and $w \in T_y(Y)$ such that $u = (v,w)$. So f is also surjective. Since the inverse of f is smooth as well, f is a diffeomorphism between $T(X \times Y)$ and $T(X) \times T(Y)$.

4. We would like to show that the tangent bundle to S^1 is diffeomorphic to $S^1 \times \mathbf{R}$. Suppose that $x = (x_1, x_2) \in S^1$, and let $x^* = (-x_2, x_1)$. Then $T_x(S^1) = \{x + tx^* | t \in \mathbf{R}\}$. Let $f : S^1 \times \mathbf{R} \to T(S^1)$ by
$$f(x,t) = (x, x + tx^*).$$
Now write $f = (f_1, f_2)$. f_1 is smooth because it is the identity on S^1. $f_2(x,t) = (x_1 - tx_2, x_2 + tx_1)$ is smooth because it is a polynomial in t, x_1 and x_2. So f is smooth. Since for any $x \in S^1$, $T_x(S^1)$ is in one-to-one correspondence with $\mathbf{R}$, f is bijective. Its inverse is $f^{-1}(y,v) = (y,t)$, where t is given by $y - v = tv^*$. Since f^{-1} is smooth as well, f is diffeomorphism from $S^1 \times \mathbf{R}$ to $T(S^1)$.

5. The projection map $\tilde{p} : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$, $\tilde{p}(x,v) = x$ is a submersion. To see this, compute the derivative:
$$d\tilde{p}_{(x,v)}(y,w) = \lim_{t\to 0} \tfrac{1}{t}\big(\tilde{p}(x + ty, v + tw) - \tilde{p}(x,v)\big) = \lim_{t\to 0}\tfrac{1}{t}\big(x + ty - x\big) = y.$$

$d\tilde{p}_{(x,v)}$ is clearly surjective at any $(x, v) \in \mathbf{R}^k \times \mathbf{R}^k$, so $\tilde{p}$ is a submersion. Suppose $\varphi : U \to X$ is a parametrization of a neighborhood of some $x \in X$ and $\Phi(x, v) = \big(\varphi(x), d\varphi_x(v)\big)$. Let us verify that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}\big(\varphi(U)\big) \arrow[r, "p"] & \varphi(U) \\
U \times \mathbf{R}^k \arrow[u, "\Phi"] \arrow[r, "\tilde{p}"'] & U \arrow[u, "\varphi"']
\end{tikzcd}$$

For any $(y, w) \in U \times \mathbf{R}^k$, there exists $(x, v) \in T(X)$ such that $\varphi(y) = x$ and $d\varphi_y(w) = v$. Then $\varphi \circ \tilde{p}(y, w) = \varphi(\varphi^{-1}(x)) = x$, and $p \circ \Phi(y, w) = p(x, v) = x$. So the diagram commutes, and since Φ is a diffeomorphism, $p = \varphi \circ \tilde{p} \circ \Phi^{-1}$ is a submersion.

6. Define a vector field $\vec{v}$ on a manifold X in $\mathbf{R}^N$ to be a smooth map $\vec{v} : X \to \mathbf{R}^N$ such that $\vec{v}(x)$ is always tangent to X at x. In other words, $\vec{v}(x) \in T_x(X)$. Hence, we can define another smooth map $\vec{v}^* : X \to T(X)$, $\vec{v}^*(x) = (x, \vec{v}(x))$. For any $x \in X$, $p \circ \vec{v}^*(x) = p(x, \vec{v}(x)) = x$. So $p \circ \vec{v}^*$ is the identity on X. Conversely, suppose that $\vec{v}^*$ is a smooth map $\vec{v}^* : X \to T(X)$ such that $p \circ \vec{v}^*$ is the identity on X. By definition, for any $x \in X$, there exist $y \in X$ and $w \in T_y(X)$ such that $\vec{v}^* = (y, w)$, so that $p \circ \vec{v}^*(y)$. But $p \circ \vec{v}^*$ is the identity, so $x = y$ and $\vec{v}^*(x) = (x, w)$ where $w \in T_x(X)$. Define $\pi : T(X) \to \mathbf{R}^N$ by $\pi(x, v) = v \in T_x(X)$, which is smooth, and $\vec{v} : X \to \mathbf{R}^N$ by $\vec{v} = \pi \circ \vec{v}^*$, which is smooth as well. Moreover, for all $x \in X$, $\vec{v}^*(x) \in T_x(X)$.
7. If k is an odd number, then there exists a non-vanishing vector field on the sphere S^k. For $k = 1$, we can let this vectorfield be $\vec{v}(x_1, x_2) = (-x_2, x_1)$, for, in fact, $(-x_2, x_1) \in T_{(x_1,x_2)}(S^1)$ as seen in exercise 4. Therefore, we can generalize to a sphere S^k by defining on it a smooth map $\vec{v}^k : S^k \to \mathbf{R}^{k+1}$ where $k+1$ is an even number, such that for any $x \in S^k$,

$$\vec{v}^k(x) = \vec{v}^k \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ \vdots \\ -x_{k+1} \\ x_k \end{pmatrix}.$$

Since $\vec{v}^k(x) \cdot x = -x_1x_2 + x_2x_1 + \cdots - x_kx_{k+1} + x_{k+1}x_k = 0$, then $\vec{v}^k(x) \in T_x(S^k)$. Therefore $\vec{v}^k$ is a non-vanishing vector field.
8. To prove that the antipodal map $\alpha : S^k \to S^k$ of a sphere S^k is homotopic to the identity if it has a non-vanishing vector field $\vec{v} : S^k \to T(X)$, it suffices to show that the statement holds whenever $|\vec{v}(x)| = 1$ on S^k. For, if there were a $y \in S^k$ such that $|\vec{v}(y)| \neq 1$, then the vector field defined for each $x \in S^k$ by $\frac{\vec{v}(x)}{|\vec{v}(x)|}$ is non-vanishing anywhere and takes on values of unit length. Now consider the smooth map $F : S^k \times \mathbf{I} \to \mathbf{R}^{k+1}$,

$$F(x, t) = x \cos \pi t + \vec{v}(x) \sin \pi t.$$

For any $x \in S^k$, we have $\langle x, \vec{v}(x) \rangle = 0$. Thus, $|x \cos t + \vec{v}(x) \sin t| = 1$ and the image of F lies in S^k. Moreover, $F(x, 0) = x$ and $F(x, 1) = -x$. So F is a homotopy between the identity on S^k and α.

9. Let the sphere bundle of a k-dimensional manifold X be defined by

$$S(X) = \{(x, v) \in T(X) \| v| = 1\}.$$

Consider the smooth map $f : T(X) \to \mathbf{R}$ defined by $f((x, v)) = |v|^2$, and let $\varphi : U \to X$ parametrize a neighborhood $\varphi(U)$ of $x \in X$. Next, define $\Phi : U \times \mathbf{R}^k \to T(X)$ by $\Phi((x, v)) = (\varphi(x), d\varphi_x(v))$ and $\tilde{f} : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$ by $\tilde{f}((x, v)) = |d\varphi_x(v)|^2$. Then the following diagram commutes:

$$\begin{tikzcd}
\Phi(U) \arrow[r, "f"] & \mathbf{R} \\
U \times \mathbf{R}^k \arrow[u, "\Phi"] \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

We claim that f has a regular value at 1. Since $\tilde{f} = f \circ \Phi$, by the Chain Rule, for any $(x, v) = \Phi((y, w)) \in T(X)$, we have $d\tilde{f}_{(y,w)} = df_{(x,v)} \circ d\Phi_{(y,w)}$. Since $d\Phi_{(x,v)}$ is an isomorphism, f has a regular value at 1 if and only if $d\tilde{f}_{(y,w)}$ is surjective for any preimage (y, w) of 1 by Φ. Hence, it suffices to show that $\tilde{f}$ has a regular value at 1. Let us compute the value of its derivative.

$$\begin{aligned}
d\tilde{f}_{(x,v)}(y, w) &= \lim_{t \to 0} \tfrac{1}{t} \left[\tilde{f}\left((x, v) + t(y, w)\right) - \tilde{f}((x, v)) \right] \\
&= \lim_{t \to 0} \tfrac{1}{t} \left[|d\varphi_{(x+ty)}(v + tw)|^2 - |d\varphi_x(v)|^2 \right] \\
&= \lim_{t \to 0} \tfrac{1}{t} \left[|d\varphi_{(x+ty)}(v)|^2 + 2t d\varphi_{(x+ty)}(v) \cdot d\varphi_{(x+ty)}(w) \right. \\
&\qquad \left. + t^2 |d\varphi_{(x+ty)}(w)|^2 - |d\varphi_x(v)|^2 \right]
\end{aligned}$$

For small t, it can be shown that $|d\varphi_{(x+ty)}(w)|$ is bounded. By definition of a derivative,

$$\lim_{s \to 0} \left(\frac{\varphi(x + ty + sw) - \varphi(x + ty)}{s} - d\varphi_{(x+ty)}(w) \right) = 0.$$

Therefore, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| d\varphi_{(x+ty)} \right| < |s|\varepsilon + |\varphi(x + ty + sw)| + |\varphi(x + ty)|$$

whenever $s < \delta$. Now φ is continuous, so for small s and t, $\varphi(x + ty + sw)$ and $\varphi(x + ty)$ are bounded. Hence $d\varphi_{(x+ty)}$ is bounded, and $\lim_{t \to 0} t |d\varphi_{(x+ty)}|^2 = 0$.

Suppose $(x, v) \in S(X) \cap U$. Since $d\varphi_x$ is surjective, for any $z \in \mathbf{R}$, we can choose $w \in \mathbf{R}^k$ such that $d\varphi_x(w) = \frac{z}{2} d\varphi_x(v)$. Now if $\tilde{f}((x, v)) = |d\varphi_x(v)|^2 = 1$, then $(0, w) \in U \times \mathbf{R}^k$ is such that $d\tilde{f}_{(x,v)}(0, w) = z$. Consequently $\tilde{f}$ and hence f have a regular value at 1. Now we invoke the Preimage Theorem to conclude that $S(X)$ is a submanifold of $T(X)$ with dimension $\dim T(X) - \dim \mathbf{R} = 2k - 1$.

10. The proof of the Whitney Immersion Theorem is basically that for the version of Whitney's Theorem presented in [1] on page 51 with the following modifications: we need only require that $N > 2k$, as well as in the induction, $M > 2k$, because we ignore the function h as well as the argument on the injectivity of the resulting function. So although we obtain an immersion in $\mathbf{R}^{2k}$, this function is not necessarily injective.
15. We claim that for every $b \in B$, there exists a neighborhood U_b of b such that $\overline{U_b} \cap A = \varnothing$. Since A and B are closed and disjoint, $X \smallsetminus A$ is open, and $B \subset X \smallsetminus A$. So for every $b \in B$, there exists a neighborhood V_b of b in X disjoint from A, and $\{V_b\}$ is an open cover for B. Also every V_b has a subset containing b which can be parametrized by a smooth map p_b defined on the unit k-ball B^k. Now the closure of $\frac{1}{2}B^k$, the open k-ball with radius 1/2, is a subset of B^k. Therefore, $U_b = p_b\left(\frac{1}{2}B^k\right)$ has closure in V_b, thus proving the claim.

    For each U_b, let us construct a smooth function $\varphi_b : X \to \mathbf{I}$ which is identically 1 on some neighborhood of b and whose support is contained in U_b. The set $\left\{\frac{1}{2}B^k, B^k \smallsetminus \overline{\frac{1}{4}B^k}\right\}$ is an open cover for B^k, so there exists a partition of unity $\{\rho_1, \rho_2\}$ subordinate to this cover. (That the partitions can be assumed to be finite will later be argued in the proof of Lemma 7, §5.) Let us assume that $\operatorname{supp}(\rho_1) \subset \frac{1}{2}B^k$. Then $\rho_1 \equiv 1$ on $\frac{1}{4}B^k$ and $\rho_1 \equiv 0$ outside $\frac{1}{2}B^k$. So let

$$\varphi_b = \begin{cases} \rho_1 \circ p_b^{-1} & \text{on } U_b \\ 0 & \text{on } X \smallsetminus U_b \end{cases}.$$

    Now let $\{\psi_i\}_{i \in \mathbf{N}}$ be a smooth partition of unity on $\bigcup U_b$ subordinate to the open cover $\{U_b\}$, and choose a $b : \mathbf{N} \to B$ to be some $b(i) \in B$ such that $\operatorname{supp} \psi_i \subset U_{b(i)}$. Then

$$\lambda = \sum_{i=0}^{\infty} \psi_i \varphi_{b(i)}$$

    is a smooth function, and $\lambda = 1$ on B.

    It remains to show that $\operatorname{supp}(\lambda)$ and A are disjoint. For any $p \in \operatorname{supp}(\lambda)$, there is a neighborhood W_p of p such that all ψ_i are identically zero on W_p except for finitely many of them, say $\psi_{i_1}, \dots \psi_{i_N}$. On W_p,

$$\lambda \equiv \sum_{j=1}^{N} \psi_{i_j} \varphi_{b(i_j)}.$$

    So $W_p \subset \bigcup_{j=1}^{N} U_{b(i_j)}$, and $U_{b(i_j)} \cap A = \varnothing$. Therefore W_p and A are disjoint. Let $W = \bigcup_{p \in \operatorname{supp}(\lambda)} W_p$. Then W is open and $W \cap A = \varnothing$. Finally, since $\operatorname{supp}(\lambda) \subset W$, $\operatorname{supp}(\lambda)$ is disjoint from A.

5. The Classification of Compact One-Manifolds

Appendix 2 of [1] offers a proof of the theorem on the classification of compact one-manifolds, whose correct statement is as follows.

Theorem 6. *Every compact connected one-dimensional manifold with boundary is diffeomorphic either to a circle or to a closed interval.*

In this section we review the proof of the Smoothing Lemma, and explain how this lemma is used to prove the Classification Theorem. We notice that there is actually a gap between what the lemma can give and what is needed to complete the proof. Then we fill in that gap. Finally we propose an alternative Smoothing Lemma, which avoids the extra step of filling a gap.

The proof of the Smoothing Lemma requires a minor correction. As defined, ρ does not guarantee that the derivative of the new function $\tilde{g}$ is everywhere positive. For

$$\tilde{g}'(x) = k\rho(x) + g'(x)(1 - \rho(x)).$$

So we require also that $0 < \rho < 1$. But then it may not always be possible that $\int_a^b \rho = 1$, especially when $b - a < 1$. We solve this problem by abandoning the latter restriction put on ρ, and instead redefine k as

$$k = \frac{\int_a^b g'(s)\rho(s)ds}{\int_a^b \rho(s)ds}.$$

This new definition allows $\tilde{g}$ to agree with g near the endpoints. If x is in a small enough neighborhood of a so that ρ is identically zero there, then

$$\tilde{g}(x) = g(a) + \int_a^x g'(s)ds = g(x),$$

and if x is in a small enough neighborhood of b, then

$$\tilde{g}(x) = g(x) + \int_a^x (k\rho(s) - g'(s)\rho(s))ds = g(x) + k\int_a^b \rho(s)ds - \int_a^b g'(s)\rho(s)ds = g(x).$$

Now we can explain how, in the paragraph before last of the proof of the Classification Theorem, the Smoothing Lemma is applied to the continuous injection $F : X \to [0, k]$ which is a diffeomorphism except perhaps at $p_1, \ldots p_{k-1}$. Now let $a_i : [i-1, i] \to [i-1, i]$ be a smooth increasing function such that $a_i^{(n)}(i) = a_i^{(n)}(i+1) = 0$ for all $n \geq 1$, and define $F^* = a \circ F$ where $a(x) = a_i(x)$ if $x \in [i-1, i]$. Then F^* is a smooth and increasing function with positive derivatives except at $p_1, \ldots p_{k-1}$. By the way F is defined, there are disjoint neighborhoods U_i of each p_i so that each $F|_{U_i} : U_i \to (i - \varepsilon_i, i + \varepsilon_i)$ is a homeomorphism. Moreover, there are smooth parametrizations of U_i which restrict to $h_i : V_i = [\varepsilon_i', \varepsilon_i''] \to U_i$, and which have smooth inverses. Then the Smoothing Lemma guarantees that for each $F^* \circ h_i : V_i \to [0, k]$, there is a smooth function $G_i : V_i \to [0, k]$ which agrees with $F^* \circ h_i$ near the endpoints. So the function $\tilde{F} : X \to [0, k]$ by

$$\tilde{F}(x) = \begin{cases} G_i \circ h_i^{-1}(x) & x \in U_i \text{ for some } i = 1, \ldots k-1, \\ F^*(x) & \text{otherwise} \end{cases}$$

is the diffeomorphism we use to replace F.

We can avoid the additional step of "smoothing" the function by including it in the lemma itself. Here is a "Better Smoothing Lemma".

Lemma 7. *Let g be a continuous function on $[a, b]$ such that it is smooth and strictly increasing on $[a, a+\varepsilon)$ and $(b-\varepsilon, b]$ for some $\varepsilon > 0$. Then there exists a smooth function $\tilde{g} : [a, b] \to \mathbf{R}$ which has positive derivative on $[a, b]$ and agrees with g in neighborhoods of a and b.*

Proof. An open cover for $[a,b]$ is $\mathcal{C} = \{[a,a+\varepsilon),(a+\delta,b-\delta),(b-\varepsilon,b]\}$ for some δ such that $0<\delta<\varepsilon$. We can assume that $\mathcal{C}$ has a finite partition of unity $\{\rho_1,\rho_2,\rho_3\}$. For if $\{\theta_i\}_{i=1}^{\infty}$ is a partition of unity subordinate to $\mathcal{C}$, we can define $A \amalg B \amalg C = \mathbf{N} \setminus \{0\}$ such that

$$\begin{aligned} A &= \{i \in \mathbf{N} | \theta_i \text{ is supported in } [a,a+\varepsilon)\} \\ B &= \{i \in \mathbf{N} | \theta_i \text{ is supported in } (a+\delta,b-\delta)\} \\ C &= \{i \in \mathbf{N} | \theta_i \text{ is supported in } (b-\varepsilon,b]\}. \end{aligned}$$

Let $\rho_1 = \sum_{i\in A}\theta_i, \rho_2 = \sum_{i\in B}\theta_i$ and $\rho_3 = \sum_{i\in C}\theta_i$. Then $\{\rho_1,\rho_2,\rho_3\}$ is a partition of unity subordinate to $\mathcal{C}$ such that ρ_1,ρ_2,ρ_3 are supported in $[a,a+\varepsilon),(a+\delta,b-\delta)$ and $(b-\varepsilon,b]$ respectively. Define

$$\tilde{g}(x) = g(a) + \int_a^x [\rho_1(t)g'(t) + k\rho_2(t) + \rho_3(t)g'(t)]dt$$

where

$$k = \frac{\int_a^b (1-\rho_1-\rho_3)(t)g'(t)dt}{\int_a^b \rho_2(t)dt}.$$

Then $\tilde{g}$ is smooth, and since $k>0$, $\tilde{g}$ is also increasing. Also, if $x \in [a,a+\delta)$,

$$\tilde{g}(x) = g(a) + \int_a^x \rho_1(t)g'(t)dt = g(x),$$

and if $x \in (b-\delta,b]$,

$$\begin{aligned} \tilde{g}(x) = g(a) + \int_a^b (\rho_1+\rho_3)(t)g'(t)dt + k\int_a^b \rho_2(t)dt - \int_x^b \rho_3(t)g'(t)dt \\ = g(a) + \int_a^b g'(t)dt - \big(g(b)-g(x)\big) = g(x). \end{aligned}$$

So $\tilde{g}$ agrees with g on $[a,a+\delta)\cup(b-\delta,b]$. □

We can now use this alternative version of the Smoothing Lemma in the paragraph before last of the proof of the Classification Theorem. We replace the continuous map F by $\tilde{F}: X \to [0,k]$, a smooth function with smooth inverse, which agrees with F on $[0,\varepsilon)\cup(k-\varepsilon,k]$ for some ε such that $0<\varepsilon<1$.

6. The Brouwer Fixed-Point Theorem

A consequence of the Classification Theorem is the Brouwer Fixed-Point Theorem, which is stated here as in [1].

Theorem 8. *(Brouwer's Fixed-Point Theorem) Any smooth map f of the closed unit ball $B^n \subset \mathbf{R}^n$ into itself must have a fixed point; that is, $f(x)=x$ for some $x \in B^n$.*

The following exercises explore other versions of the Brouwer Theorem and culminate with a concrete application to linear algebra.

2. The smooth complex function $\varphi : B^2 \to \mathbf{C}$ discussed in [2] and defined by

$$\varphi(z) = i\frac{1-z}{1+z} = \frac{i(1-x)+y}{1+x+iy} = \frac{\big(i(1-x)+y\big)\big(1+x-iy\big)}{(1+x)^2+y^2} = \frac{2y}{(1+x)^2+y^2} + i\frac{1-x^2-y^2}{(1+x)^2+y^2}$$

maps into $H^2 \cup \{\infty\}$ because $1-x^2-y^2 \geq 0$ for any $z \in B^2$. We can also check that φ has a smooth inverse

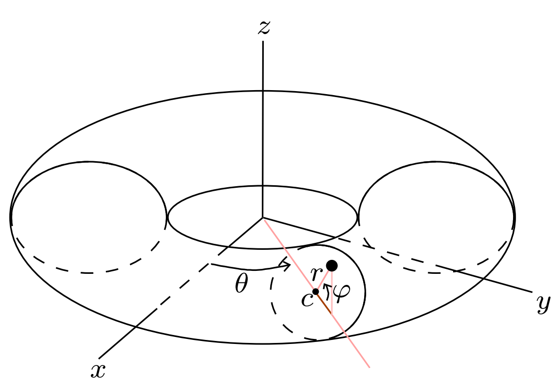
$$\varphi^{-1}(z) = \frac{i-z}{i+z}$$

which maps into B^2 since $|i-z| \leq |i+z|$ for any $z \in H^2 \cup \{\infty\}$. Let $f : \mathbf{C} \to \mathbf{C}$ by $f(z) = z+1$. Then f has a unique fixed point at ∞. Since $\varphi(-1) = \infty$, -1 is the only fixed point of $g = \varphi^{-1} \circ f \circ \varphi : B^2 \to B^2$. Now if we associate each point $x+iy$ in $\mathbf{C}$ with a point (x,y) in $\mathbf{R}^2$, we obtain a diffeomorphism $\tilde{g} : B^2 \to B^2$ with a unique fixed point at $-1 \in \partial B^2$.

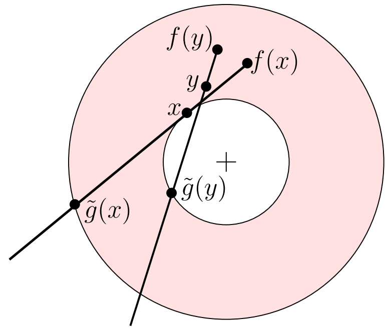
Now the smooth function does not necessarily have to be a diffeomorphism. Any constant function $h : B^n \to B^n$ mapping to a point $z \in \partial B^n$ clearly has a unique fixed point on the boundary.

3. Suppose that a solid ring torus T is situated in $\mathbf{R}^3$ in such a way that the z-axis is its axis of symmetry and that the xy-plane is its plane of symmetry. Then a rotation around the z-axis has no fixed point. Also, define a coordinate system on T so that any point $v \in T$ can be described by three quantities (θ, φ, r), where
   - θ is the angle between the positive x-axis and the projection v' of v onto the xy-plane,
   - φ is the angle between the vector pointing away from the origin, whose tail lies in the center c of the disc created by intersecting the plane containing the origin, v and v' with T, and the vector R whose head lies at v and tail at c,
   - r is the norm of R.

   Then the map $(\theta, \varphi, r) \mapsto (\theta + t, \varphi + u, r)$ for some $t \in (0, 2\pi)$ and some $u \in \mathbf{R}$ has no fixed point.



The proof of the Brouwer Theorem fails when we try to construct a retraction $\tilde{g} : T \to \partial T$. The natural way to extend the definition of g to a torus is to let $\tilde{g}(x)$ be the point where the line segment starting at $f(x)$ and passing through x first hits the boundary. Although the restriction of $\tilde{g}$ to ∂T is the identity, since T is not star-shaped around any point, $\tilde{g}$ is not necessarily continuous, hence not a retraction.



4. Let $f : B_a = \{x \in \mathbf{R}^k | |x|^2 < a\} \to \mathbf{R}^k$ by
$$f(x) = \frac{ax}{\sqrt{a^2 - |x|^2}}.$$
f is clearly smooth on B_a. Its inverse,
$$f^{-1}(x) = \frac{ax}{\sqrt{a^2 + |x|^2}},$$
is also smooth on all of $\mathbf{R}^k$ and maps into B_a. Therefore f is a diffeomorphism between B_a and $\mathbf{R}^k$. The translation $g : \mathbf{R}^k \to \mathbf{R}^k$ by $g(x) = x + 1$ has no fixed points, so the smooth map $f^{-1} \circ g \circ f : B_a \to B_a$ has no fixed points as well.
6. Let $f : B^n \to B^n$ be a continuous function from the closed n-ball into itself. For any ε such that $0 < \varepsilon < 2$, choose $\delta > 0$ small enough so that $2\delta/(1+\delta) < \varepsilon$. By the Weierstrass Approximation Theorem, there exists a polynomial $\tilde{p} : \mathbf{R}^n \to \mathbf{R}^n$ such that $|\tilde{p} - f| < \delta$ on B^n. Therefore $|\tilde{p}| < 1 + \delta$ on B^n. Now define $p : B^n \to \mathbf{R}^n$ by $p = \tilde{p}/(1+\delta)$. Then $|p| < 1$, so p maps to a subset of B^n. Moreover,
$$|p - f| = \left| \frac{\tilde{p}}{1+\delta} - f \right| < \frac{2\delta}{1+\delta} < \varepsilon.$$
So given $\varepsilon > 0$, we have found a polynomial $p : B^n \to B^n$ such that $|f - p| < \varepsilon$. If f has no fixed points, then there exists $c > 0$ such that for all $x \in B^n$, $|f(x) - x| > c$. Choose p so that $|f - p| < c$. By the Brouwer Fixed-Point Theorem, p has a fixed point $y \in B^n$. So $|f(y) - y| = |f(y) - p(y)| < c$, which is a contradiction. So f cannot have a fixed point.
7. For the proof of the following theorem, we will use the Brouwer Theorem for continuous functions $f : B^{n-1} \to B^{n-1}$, which was proven in the previous exercise.

Theorem 9. *(Frobenius' Theorem) If the entries of an $n \times n$ real matrix A are all nonnegative, then A has a real nonnegative eigenvalue.*

Proof. If A is singular, then there is a nonzero vector $v \in \operatorname{Ker} A$. So $Av = 0 = 0v$ and 0 is an eigenvalue. Therefore, we may assume that A is nonsingular. Denote the entries of A by a_{ij}, and consider the smooth map $f : S^{n-1} \to S^{n-1}$ by $f(v) = Av/|Av|$. For any x in the "first quadrant" $Q = \{(x_1, \dots x_n) \in S^{n-1} | \text{ all } x_i \geq 0\} \smallsetminus \{0\}$,
$$(f(x))_i = \left(\frac{Ax}{|Ax|} \right)_i = \frac{1}{|Ax|} \sum_{j=1}^{n} a_{ij} x_j \geq 0.$$

So f maps Q into itself. Now let $\varphi : Q \to B^{n-1}$ be a homeomorphism to the closed unit $n-1$-ball. By the Brouwer Theorem applied to the continuous map $\varphi \circ f \circ \varphi^{-1} : B^{n-1} \to B^{n-1}$, there exists a vector $x \in B^{n-1}$ such that $\varphi \circ f \circ \varphi^{-1}(x) = x$ or

$$A(\varphi^{-1}(x)) = |A(\varphi^{-1}(x))|\varphi^{-1}(x).$$

Therefore $|A(\varphi^{-1}(x))|$ is a positive eigenvalue for A. $\square$

7. Conclusion

Guillemin and Pollack's book is an excellent introduction to differential topology for any student who has studied linear algebra, calculus (Inverse Function Theorem), and basic topology (connectedness, compactness). Full of illustrations throughout, the book is set up in a way which encourages the development of intuition. The definition of a manifold, given in $\mathbf{R}^N$, contributes to this effort. Although manifolds could have been defined in a more abstract setting, namely in an arbitrary topological space, the choice made by the authors leads more quickly to important results. The style in which the proofs are written is loose, sometimes even incomplete, as it would seem to the beginning student in mathematics. However, constantly questioning the proofs and filling up missing steps naturally evolve into habits, thus providing for a good exercise in mathematical thinking. In this respect, all the problems proposed at the end of each section are worth considering. While some are simply examples illustrating definitions or counterexamples to a theorem, others explore applications to results discussed earlier and variants to proofs. The student should be aware, however, that certain concepts in the book do not agree with standard definitions. In particular, the tangent space of a manifold $X \subset \mathbf{R}^N$ at a point x is usually taken to be a subset of $T(X)$ and not of $\mathbf{R}^N$.

Given that one's intuition is often responsible for the birth of an idea which will solve a problem, and that drawing pictures, when not provoking the puzzlement of a non mathematically trained friend who has peaked over the shoulder, marks the beginning of a strengthening idea, differential topology is undoubtedly really enjoyable to learn. Needless to say, such a visual subject must be accompanied with some difficulty of mastery. Quite a bit of work is required in order to understand the formal definition for a manifold. The trouble lies mainly in the task of turning a loose idea into a rigorous proof. Sometimes, intuition does fail at guiding one's thought in the correct direction. But in the end, one is always rewarded for having studied this delightful, and yet demanding subject. Such is how I feel. I would like to thank Professor Mess for having introduced and guided me through this challenge.

References

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